Explicit analytical solutions of the coupled differential equations for porous material drying*

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Abstract Some explicit analytical solutions are derived for the coupled partial differential equation set describing porous material drying with two extraordinary methods proposed by the authors, i.e. the method of separating variables by addition and the method of evaluating the source term in reverse order. Besides their theoretical meaning, these solutions can also be the standard solutions for the computational solutions of heat and mass transfer.

Keywords: **coupled heat and mass transfer, analytical solution, drying, porous material.**

Porous material drying is a typical coupled heat and mass transfer process. In order to raise the drying rate, it is necessary to investigate the coupled effect of heat and mass transfer. A governing equation set coupled heat and mass transfer was given by ref. [1] based on Luikov equation and a simplified model shown in fig. 1. Assumptions for the above-mentioned equation set are as follows.

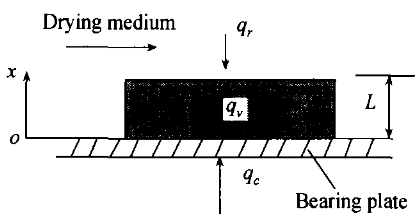

Fig. 1. Physical model of governing equation set.

(i) The material is an infinite plate with thickness L; the heat and mass transfer is only one-dimensional along the thickness direction x; there are internal heat source q_v in the plate and supplementary heating q_r and q_c on and beneath the plate. (ii) The skeleton of the porous material does not deform in the drying process. (iii) Since the construction of porous material is fine and close, the migration of humid content due to gravity and pressure gradient and the heat transfer due to humid content migration are neglected. (iv) The humid content in the porous material and the skeleton of the porous material are under thermodynamic equilibrium condition; the coefficients (for example, thermodynamic property coefficients) in the governing equation set are constants. (v) The external drying conditions are constant. The governing equation set is shown as follows:

$$\frac{\partial T}{\partial \tau} = a_e \frac{\partial^2 T}{\partial x^2} + \frac{\varepsilon h_{fg} C_m}{C_s} \frac{\partial \theta}{\partial \tau} + \frac{q_v}{\rho C_s}, \tag{1}$$

$$\frac{\partial \theta}{\partial \tau} = D_m \frac{\partial^2 \theta}{\partial x^2} + \frac{D_m \delta}{C_m} \frac{\partial^2 T}{\partial x^2}, \tag{2}$$

where T is temperature, τ time coordinate, a_e thermal diffusion coefficient, ε phase change factor,

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h_{fg} evaporating latent heat, θ mass transfer potential, $C_m = (\partial u/\partial \theta)_T$, u humid content based on dry porous material, ρ density, D_m mass diffusion coefficient, δ heat gradient coefficient, C_s specific heat.

The initial and boundary conditions given by ref. [1] are as follows:

$$\tau = 0,\ T = T_0 = \text{Const.}, \tag{3}$$

$$\theta = \theta_0 = \text{Const.}; \tag{4}$$

$$x = 0,\ -\lambda_e \frac{\partial T}{\partial x} = q_c, \tag{5}$$

$$C_m \frac{\partial \theta}{\partial x} + \delta \frac{\partial T}{\partial x} = 0; \tag{6}$$

$$x = L,\ q_r + \alpha_q (T_s - T) = \lambda_e \frac{\partial T}{\partial x} + (1 - \varepsilon)\alpha_m h_{fg}(\theta - \theta_e), \tag{7}$$

$$-D_m \rho_s \left(C_m \frac{\partial \theta}{\partial x} + \delta \frac{\partial T}{\partial x} \right) = \alpha_m (\theta - \theta_e), \tag{8}$$

where α_q and α_m are the connective heat and mass transfer coefficients from the plate to the surrounding, T_s drying medium temperature, θ_e mass transfer potential of drying medium, λ_e heat conduction coefficient.

Some analytical solutions of the above-mentioned governing equation set are derived to enrich the drying theory. These solutions can also be the standard solutions for the computational heat and mass transfer. Since the main aim is to obtain the explicit solutions of the governing equation set, the initial and boundary conditions are indeterminate before derivation and then deduced from the solutions afterward, which is similar to the derivation of the classical analytical solutions for incompressible flow in the last century. For the same reason, the above-mentioned fifth assumption of ref. [1] is unnecessary, i.e. the initial and boundary conditions (3)—(8) can be partially varied, which provided they are physically reasonable. Some terms such as θ_0, T_0, α_m and α_q are not necessarily constants. Actually, they are the conditions often occurred in practice.

1 An analytical solution derived with the method of separating variables by addition

An extraordinary method was proposed by the authors[2,3] to derive analytical solutions of some heat transfer partial differential equations, i.e. instead of the ordinary method of separating variables by assuming $\theta(\tau, x) = W(\tau)\cdot X(x)$, this method assumes $\theta(\tau, x) = W(\tau) + X(x)$. It is found that the latter (perhaps can be named the method of separating variables by addition) is able to solve many partial differential equations which can not be solved by the ordinary method of separating variables. This extraordinary method is developed here to solve the above-mentioned governing equation set with two unknown functions.

Rewriting eqs. (1) and (2) as follows to simplify the ensuing paragraphs:

$$\frac{\partial T}{\partial \tau} = a_e \frac{\partial^2 T}{\partial x^2} + B \frac{\partial \theta}{\partial \tau} + Q, \tag{1a}$$

$$\frac{\partial \theta}{\partial \tau} = D_m \frac{\partial^2 \theta}{\partial x^2} + E \frac{\partial^2 T}{\partial x^2}, \tag{2a}$$

where $B = \varepsilon h_{fg} C_m / C_s$, $Q = q_v / (\rho C_s)$, and $E = D_m \delta / C_m$.

It is assumed that the unknown functions $T = W_1(\tau) + X_1(x)$, $\theta = W_2(\tau) + X_2(x)$; and $Q = W_3(\tau) + X_3(x)$ is a known function. Then, eqs. (1a) and (2a) can be expressed as

$$W'_1 = a_e X''_1 + BW'_2 + W_3 + X_3, \tag{1b}$$

$$W'_2 = D_m X''_2 + EX''_1. \tag{2b}$$

Rearranging these two expressions and considering the equality of functions with different independent variables equivalent to a constant, it is derived:

$$W'_1 - BW'_2 - W_3 = C_3 = a_e X''_1 + X_3, \tag{9}$$

$$W'_2 = C_1 = D_m X''_2 + EX''_1. \tag{10}$$

From the left side of eq. (10), it is deduced:

$$W_2 = C_1 \tau + C_2. \tag{11}$$

Substituting eq. (11) into the left side of eq. (9), it is obtained:

$$W_1 = (C_1 B + C_3)\tau + \int W_3(\tau) d\tau. \tag{12}$$

From the right side of eq. (9), it is deduced:

$$X_1 = C_3 x^2/(2a_e) + C_4 x + C_5 - \iint X_3(x) dx dx / a_e. \tag{13}$$

Substituting eq. (13) into the right side of eq. (10), it is obtained:

$$X_2 = \left[(C_1 - C_3 E/a_e) x^2/2 + C_6 x + E \iint X_3(x) dx dx / a_e \right] \Big/ D_m. \tag{14}$$

Therefore, an analytical solution of equation set (1) and (2) is derived as follows using the method of separating variables by addition:

$$T = (C_1 B + C_3)\tau + \int W_3(\tau) d\tau + C_3 x^2/(2a_e) + C_4 x + C_5 - \iint X_3(x) dx dx / a_e, \tag{15}$$

$$\theta = C_1 \tau + C_2 + \left[(C_1 - C_3 E/a_e) x^2/2 + C_6 x + E \iint X_3(x) dx dx / a_e \right] \Big/ D_m. \tag{16}$$

In order to investigate the initial and boundary conditions of eqs. (15) and (16), a simplified case is chosen: $X_3(x) = 0 = W_3(\tau)$, i.e., $q_v = 0$; $C_3 = C_1 a_e / E$, $C_6 = -C_4\delta / C_m$. Then the analytical solution can be rewritten as

$$T = (B + a_e/E)C_1\tau + C_1x^2/(2E) + C_4x + C_5, \tag{17}$$

$$\theta = C_1\tau + C_2 - C_4\delta x/C_m. \tag{18}$$

Therefore, the boundary conditions at $x = 0$ can be expressed as

$$q_c = -\lambda_e \frac{\partial T}{\partial x} = -\lambda_e(C_1x/E + C_4) = -C_4\lambda_e = \text{Const.}, \tag{19}$$

$$C_m \frac{\partial \theta}{\partial x} + \delta \frac{\partial T}{\partial x} = 0 = C_m(-C_4\delta/C_m) + \delta(C_1x/E + C_4). \tag{20}$$

It can be concluded from these two expressions that the above-mentioned boundary condition (6) is satisfied, and the boundary condition (5) is equivalent to a constant heat transfer q_c at the lower boundary.

One of the boundary conditions at $x = L$ can be expressed as

$$-D_m\rho_s[C_m(-C_4\delta/C_m) + \delta(C_1L/E + C_4)] = \alpha_m(C_1\tau + C_2 - C_4\delta L/C_m - \theta_e),$$

then

$$\alpha_m = -C_1\rho_s C_m L/(C_1\tau + C_2 - C_4\delta L/C_m - \theta_e). \tag{21}$$

It means that the external connective mass transfer coefficient on upper boundary is not a constant but a function of time. This condition does not agree with the assumption given by ref. [1], but it is physically possible.

Another boundary condition at $x = L$ can be expressed as

$$q_r = \lambda_e(C_1L/E + C_4) - (1 - \varepsilon)h_{fg}C_1C_m\rho_sL$$
$$- \alpha_q[T_s - (B + a_e/E)C_1\tau - C_1L^2/(2E) - C_4L - C_5]. \tag{22}$$

It means that the heat transfer on the upper boundary is not a constant but a function of time also. Nevertheless, it is physically possible too.

The initial conditions at $\tau = 0$ are expressed as

$$T = C_1x^2/(2E) + C_4x + C_5, \tag{23}$$

$$\theta = -C_4\delta x/C_m + C_2. \tag{24}$$

It is found that the temperature and mass tranfer potential are not even along the thickness direction at initial time, which is different from ref. [1] but physically possible also.

Therefore, an explicit analytical solution is derived for the coupled partial differential equation set describing porous material drying using the method of separating variables by addition.

2 Another set of possible analytical solutions

Actually, there are three dependent variables (T, θ and Q) in the governing equation set (1a) and (2a). If the source term Q is a given function, for example, $Q = W_3(\tau) + X_3(x)$ or $Q = 0$, then to derive analytical solutions is a hard job or even impossible. However, if the aim is only to obtain more analytical solutions, another possible way is not to fix Q beforehand, but to evaluate its expression finally. Then, it is possible to find out infinite number of analytical solutions for the development of theory and application in computational heat transfer. Perhaps this method can be named the method of evaluating source term in the reverse order.

For a given arbitrary function $\theta = \theta(\tau, x)$, $\frac{\partial \theta}{\partial \tau}$ and $\frac{\partial^2 \theta}{\partial x^2}$ can be derived. Therefore, $T(\tau, x)$ can be deduced from eq. (2a).

$$T = \iint \left(\frac{\partial \theta}{\partial \tau} - D_m \frac{\partial^2 \theta}{\partial x^2} \right) dx dx / E \tag{25}$$

Then the function $Q(\tau, x)$ can be obtained from eq. (1a):

$$Q = \frac{\partial}{\partial \tau} \iint \left(\frac{\partial \theta}{\partial \tau} - D_m \frac{\partial^2 \theta}{\partial x^2} \right) dx dx / E - a_e \left(\frac{\partial \theta}{\partial \tau} - D_m \frac{\partial^2 \theta}{\partial x^2} \right) \Big/ E - B \frac{\partial \theta}{\partial \tau} \tag{26}$$

For example, if $\theta = C_3 e^{C_1 \tau + C_2 x}$, then it is obtained: $T = C_3 (C_1 - C_2^2 D_m) e^{C_1 \tau + C_2 x} / (C_2^2 E)$ and $Q = C_3 [(C_1 - a_e)(C_1 - C_2^2 D_m) - C_1 C_2^2 BE] e^{C_1 \tau + C_2 x} / C_2^2 E$. So, with different given analytical functions, $\theta(\tau, x)$, there will be corresponding explicit analytical solutions $T(\tau, x)$ and $Q(\tau, x)$ provided eqs. (25) and (26) are integrable. Therefore, it is able to derive infinite number of analytical solutions. Of course, their initial and boundary conditions can be evaluated similarly to that described in the preceding paragraph. By the way, the example given in this passage can satisfy the condition $Q = 0$, which provided that the constants C_1 and C_2 fulfill $(C_1 - a_e)(C_1 - C_2^2 D_m) - C_1 C_2^2 BE = 0$. It is an algebraic equation of second order and is easy to solve.

3 Concluding remark

Two algebraically explicit analytical solution families are derived using two extraordinary methods for the coupled partial differential equation set describing porous material drying. Actually, there are infinite number of solutions since both solution families include arbitrary functions. According to the knowledge of the authors, no such analytical solutions have been published in the open literatures.

It is impossible to apply these solutions to solve arbitrary practical problems since they may not satisfy arbitrary initial and boundary conditions. However, the theoretical meaning of analytical solutions is irreplaceable, they can absolutely accurately describe the situations reflected by the governing equations under given initial and boundary conditions. In addition, as standard solutions, these analytical solutions are useful to the computational heat and mass transfer.

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